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On the separation of fast and slow motions in mechanical systems with high-frequency modulation of the dissipation coefficient

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A R T I C L E I N F O

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A B S T R A C T

Dynamics of a model mechanical system with 'fast and strong' oscillations of the damping coefficient has been analyzed by Fidlin (2005) [6]. He has performed the asymptotic analysis of the equation of motion of this system to conclude that these oscillations produce variation in its effective stiffness. The present paper continues analysis of dynamics of the system in the regimes of motion treated by Fidlin as well as in those left out in his asymptotic solution. The results are compared with the results of the solution of the classical Mathieu equation, which features fast oscillations in the stiffness of a system. The influence of stiffness and damping modulations on the stability of motion of corresponding oscillators is studied. Several engineering applications modeled by the system with oscillations of the damping coefficient are introduced. Analysis of motion of this system exemplifies how the method of direct separation of motions (Blekhman (2000) [7]) can be applied for solving equations with fast oscillating terms depending on the velocities. Some features of the application of the method of direct separation of motions in this case and in the similar ones are highlighted.

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1. Introduction

Oscillators with one degree of freedom (a linear oscillator, nonlinear oscillators—the Van der Pol oscillator, the Duffing oscillator, the Mathieu oscillator and others) are the generic models in the modern theory of oscillations (see e.g. [1–4]). Many papers are devoted to analysis of their behavior. Various mechanical systems and effects can be described and studied with their help. To this end, it is rather surprising that the one-degree-of-freedom mechanical system with modulation of the dissipation coefficient has not yet gained much attention in the literature. It has been introduced in the paper [5] by Fidlin. We believe that this mechanical system is as generic as those already listed, and we call it hereafter the Fidlin oscillator.

In monograph Fidlin [6] devotes much attention to the systems with strong general high frequency excitation, which can be described by the equation

$$\ddot{x} = F(\dot{x}, x, t, \tau) + \omega\Phi(\dot{x}, x, t, \tau)$$

where the dot indicates total differentiation to time t, $x|_{t=0}=x_0$, $\dot{x}|_{t=0} = v_0$, $x_0=O(1)$, $v_0=O(1)$, $\tau=\omega t$ is 'fast' time, and the frequency ω is much larger than one.

The general mathematical approach to the asymptotic analysis of motion of mechanical systems with strong high frequency excitation has been devised by Fidlin. This asymptotic approach, based on the generalized averaging, is relatively cumbersome and not transparent for its physical interpretation.

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As an illustration, Fidlin has examined the case of strong high frequency oscillation of the dissipation coefficient, i.e. the following equation:

$$\ddot{x}+\beta\dot{x}+x=a\omega\dot{x}\cos\omega t \tag{1}$$

He has emphasized that this example is purely mathematical.

Eq. (1) has been asymptotically solved assuming that the parameters a and β are of order one, whereas the frequency is much larger than one [5]. As a result, the following equation of 'slow' motion has been obtained:

$$\ddot{X}+\beta\dot{X}+I_0^2(a)X=0 \tag{2}$$

Here $X=\langle x\rangle$—is the 'slow' component of motion, $\langle\ldots\rangle$ designates averaging in the period 2π on fast time variable $\tau=\omega t$, $I_0(a)$ is the modified Bessel function. According to [6], solutions to Eqs. (1) and (2) are asymptotically close to each other for the time interval $t=O(1)$. In Fidlin assumptions the point $x=0$ is always stable for Eq. (1), so x and $\dot{x}$ are bounded.

Eq. (2) is remarkable in two aspects: 1) under adopted assumptions on the orders of magnitudes of the parameters involved, the zero solution $X=0$ is inherently stable. 2) Eq. (2) characterises variation in the effective stiffness of the system generated by pulsations of the damping coefficient in the original system (1).

In the present paper, the analysis of the Fidlin oscillator is extended to a broader range of parameters by means of the concept of vibrational mechanics and the method of direct separation of motions [7,8] (see also [9–11]). It is worth to mention, that Fidlin believes that this method is not applicable for solving problems concerned with the strong high frequency velocity-dependent excitation, in particular, for solving Eq. (1). Authors of the present paper show that the method of direct separation of motions can indeed be applied for solving such problems. Thereby, the range of applicability of the method of direct separation of motions is extended.

In the considered broader range of parameters, the behavior of the solution of the equation of the Fidlin oscillator is compared with the behavior of the solution of the classical equation with oscillating stiffness. Several engineering applications modeled by the equation of the Fidlin oscillator are introduced. We note that the search of these applications has been rather challenging. All applied problems, analyzed before [7,8], did not require to account for the dependence of fast component on slow time, as is necessary in the case of the Fidlin oscillator. Fidlin himself has believed that the equation he has examined is interesting only from the mathematical point of view. We think that it is not the case: several applications, although rather exotic, have already been found, and more examples will certainly appear. We also draw the attention to the remarkable, and not yet completely understood from the physical point of view, fact, that the modulation of the dissipation coefficient produces the increase of the effective stiffness of the system with respect to 'slow' loads.

Vibrational mechanics is mechanics, which should be used by an observer, who do not notice "fast forces and fast motions". This observer should add so-called vibrational forces, which are calculated following certain rules, to all slow forces exerted on a system. To this end, vibrational mechanics is similar to the mechanics of the relative motion. The "fast motions" are ignored in the vibrational mechanics in the same manner as the relative motion of a system. Naturally, the condition that forces exerted on a system and its motions can be separated on fast and slow components has to be fulfilled for the applicability of such a conception. Mathematical formalization of this assumption and matters of its correctness are reported in the books [7,8]. General conception of neglecting motions, including the dynamics of relative motion, is stated in the same books. The method of direct separation of motions is efficient for obtaining expressions for vibrational forces and compiling the main equations of vibrational mechanics. Our description implies that the reader either have read the books [7,8], or can do so if he or she wishes. Otherwise, the present paper would become too long.

The concept of vibrational mechanics and the method of direct separation of motions facilitate solution of various challenging problems of action of high frequency vibrations on nonlinear mechanical systems [7,8]. Distinctive features of this concept and this method are the simplicity in application and the transparency of the physical interpretation. It is also remarkable that approximations are involved only for solving equations of 'fast' motions. These approximations do not strongly affect the accuracy of solving equations of 'slow' motions, because only averaged fast components are employed in their formulation.

Solving Eq. (1) by the method of direct separation of motions we assume, that its solution can be written as

$$x=X(t)+\psi(t,\omega t) \tag{3}$$

where X—'slow', and ψ—'fast', 2π—periodic in dimensionless ('fast') time $\tau=\omega t$ variable, with average zero:

$$\langle\psi(t,\tau)\rangle=0 \tag{4}$$

where for any $h=h(t,\tau)$, T—periodic in τ, we define $\langle h(t,\tau)\rangle=\frac{1}{T}\int_0^T h\,d\tau$.

The assumption that solution of the initial equation can be sought as the sum (3) of 'slow' and 'fast' components is the fundamental assumption of vibrational mechanics. It has been mathematically formalized in books [7,8]. In Sections 2.2 and 3, verification of this assumption is provided for the considered Eq. (1).

Following the method of direct separation of motions, we obtain equations of 'slow' and 'fast' motions in the form:

$$\ddot{X}+\beta\dot{X}+X=a\omega\langle\dot{\psi}\cos\omega t\rangle \tag{5}$$

$$\ddot{\psi}+\beta\dot{\psi}+\psi=a\omega(\dot{X}\cos\omega t+\dot{\psi}\cos\omega t-\langle\dot{\psi}\cos\omega t\rangle) \tag{6}$$

Eq. (5) is obtained by the averaging of Eq. (1) in period $2\pi/\omega$, Eq. (6)—from the condition of the correctness of the initial Eq. (1) with expression (3) taken into account.

2. Solving the equation of Fidlin oscillator by the method of direct separation of motions

2.1. Conventional approximate solution

Here we will solve Eq. (1) assuming, as Fidlin, that the parameters a and β are of order one, whereas the frequency is much larger than one $\omega \gg 1$.

In this section, we use the method of direct separation of motions in its conventional form [7,8], i.e. we solve equation of 'fast' motion (6) approximately, using all standard assumptions. The principle one is the following: while solving the equation of 'fast' motion, it is possible to consider all involved 'slow' variables as constants ('frozen').

The periodic solution of Eq. (6) has the form of an infinite series:

$$\psi = B_1 \cos\omega t + B_2 \sin\omega t + C_1 \cos 2\omega t + C_2 \sin 2\omega t + \cdots \tag{7}$$

The constant term is missing in solution (7) because ψ is 'fast', 2π—periodic in dimensionless ('fast') time $\tau = \omega t$ variable, with average zero (4).

In solution (7) we take into account only the first two terms. It will be shown later, that it is justified for $a < 1$. Consequently for B_1 and B_2 we get following expressions:

$$B_1 = \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2\beta^2} a\omega\dot{X}, \quad B_2 = \frac{\omega\beta}{(1-\omega^2)^2+\omega^2\beta^2} a\omega\dot{X} \tag{8}$$

With $\omega \gg 1$, formulas (8) take the form:

$$B_1 = -\frac{a}{\omega}\dot{X}, \quad B_2 = \frac{a\beta}{\omega^2}\dot{X} \tag{9}$$

To derive from (5) the equation of 'slow' motion, it is sufficient to determine the function $a\omega\langle\dot{\psi}\cos\omega t\rangle$. Using formulas (7), (9) and taking into account that $\langle \sin 2\omega t\rangle = 0$, $\langle \cos^2 \omega t\rangle = 1/2$, we find:

$$a\omega\langle\dot{\psi}\cos\omega t\rangle = \frac{a^2\beta}{2}\dot{X} \tag{10}$$

According to this, the equation of 'slow' motion (5) may be written as:

$$\ddot{X} + \beta\left(1-\frac{a^2}{2}\right)\dot{X} + X = 0 \tag{11}$$

As is seen, the equation of 'slow' motion (11) differs substantially from Eq. (2) obtained by Fidlin [5].

Thus, applying the method of direct separation of motions in its conventional form for the solution of Eq. (1), we have obtained the incorrect equation of 'slow' motion. Therefore, we may conclude that either the method of direct separation of motions cannot be applied for solving the Eq. (1) at all, or it can be applied, but without one or several assumptions, usually introduced while solving equations by this method.

2.2. Assessment of validity of the main assumption of vibrational mechanics

In this section we conduct the verification of the possibility of applying the method of direct separation of motions for the solution of the Eq. (1). This verification is a posteriori, i.e. the solution obtained before in Section 2.1 is used in it. Should the inaccuracy of this solution be the result of the impossibility of applying the method of direct separation of motions in the case of the Eq. (1), it would be revealed.

The method of direct separation of motions can be applied if the main assumption of vibrational mechanics—the assumption that solutions of initial equation have the form (3), is fulfilled. Practically we verify, whether the variable X is "indeed slow" as compared with the variable ψ. For this purpose, accordingly to [7], we introduce amplitudes X_0 and ψ_0 of the components X and ψ:

$$X/X_0 = O(1), \quad \psi/\psi_0 = O(1)$$

We regard the component X as being slow as compared to ψ if the following condition is held $(\dot{X}/X_0)/(\dot{\psi}/\psi_0) = O(\varepsilon)$, where ε—small parameter.

Using derived in Section 2.1. expressions for B_1 and B_2 we obtain: $\dot{X} = O(k)$, $\psi_0/X_0 = O((a/\omega)k)$, $\dot{\psi} = O(ak)$, where $k(a, \beta)$—is the frequency of 'slow' free vibrations, which can be determined from the resulting equation of 'slow' motions. So in our case:

$$\frac{\dot{X}}{X_0}\psi_0/\dot{\psi} = O\left(\frac{k}{\omega}\right) \tag{12}$$

From Eq. (11) we obtain the following expression for this frequency:

$$k = \sqrt{1-\frac{\beta^2}{4}\left(1-\frac{a^2}{2}\right)^2} < 1 \tag{13}$$

Since the parameters a and β are of order one the main assumption of vibrational mechanics can be written in the form:

$$\frac{1}{\omega} < \varepsilon \ll 1 \tag{14}$$

The condition (14) is always fulfilled, because the frequency ω is much larger than one.

For large magnitudes of the parameter a, when $a > \sqrt{(4/\beta)+2}$, value of the frequency of 'slow' free vibrations $k(a, \beta)$ becomes complex, and the 'slow' motion in this case is not oscillatory. Hence the main assumption of vibrational mechanics fulfils automatically.

Therefore, the method of direct separation of motions can be applied for solving the Eq. (1).

2.3. More accurate solution with dependence of 'fast' motion on 'slow' time taken into account

As has been already mentioned, one of the principal assumptions, usually introduced while solving equations by the method of direct separation of motions, is the following: while solving the equation of 'fast' motion, it is possible to consider all involved 'slow' variables as constants ('frozen'). Here we conduct the solution of Eq. (1) without employing this assumption.

As before, while solving Eq. (6) we take into account only the first two terms in series (7), which is justified for $a < 1$. Now in Eq. (6) we do not consider 'slow' velocity $\dot{X}$ as constant ('frozen'), so B_1 and B_2—are some slow-time t depending functions. Then we obtain for $\dot{\psi}$ and $\ddot{\psi}$

$$\dot{\psi} = (\dot{B}_1 + \omega B_2)\cos\omega t - \omega B_1 \sin\omega t \tag{15}$$

$$\ddot{\psi} = -\omega^2 B_1 \cos\omega t + (-2\omega\dot{B}_1 - \omega^2 B_2)\sin\omega t \tag{16}$$

In formulas (15) and (16), we have neglected asymptotically small terms. Their orders of magnitude were assessed using the expressions (9), i.e. $B_1 = O(1/\omega)$, $B_2 = O(1/\omega^2)$, where $\omega \gg 1$.

Thus, from the equation of 'fast' motions we have obtained two equations: one for terms with factor $\cos\omega t$, another—with factor $\sin\omega t$:

$$-\omega^2 B_1 = a\omega\dot{X} \tag{17}$$

$$-2\omega\dot{B}_1 - \omega^2 B_2 - \omega\beta B_1 = 0 \tag{18}$$

In Eqs. (17) and (18) we have neglected asymptotically small terms. Solving these equations, we get

$$B_1 = -\frac{a}{\omega}\dot{X} \tag{19}$$

$$B_2 = \frac{a}{\omega^2}(2\ddot{X} + \beta\dot{X}) \tag{20}$$

Using the equality

$$\langle \dot{\psi}\cos\omega t \rangle = \frac{1}{2}(\dot{B}_1 + \omega B_2) \tag{21}$$

we obtain

$$\ddot{X} + \beta\dot{X} + X = \frac{a\omega}{2}(\dot{B}_1 + \omega B_2)$$

Now, employing formulas (19) and (20) we get equation of 'slow' motions in the form:

$$\left(1 - \frac{a^2}{2}\right)\ddot{X} + \beta\left(1 - \frac{a^2}{2}\right)\dot{X} + X = 0 \tag{22}$$

It also can be written as

$$\ddot{X} + \beta\dot{X} + \frac{2}{2 - a^2}X = 0 \tag{23}$$

The obtained equation of 'slow' motion is in a good agreement with Eq. (2) for $a < 1$. Moreover, Taylor series expansion of the function $I_0^2(a)$ with order a^2 terms retained yields $I_0^2(a) = 1 + (a^2/2) + O(a^4)$. The X coefficient in Eq. (23) is exactly the same $1 + (a^2/2) + O(a^4)$.

Thus, solving Eq. (1) by the method of direct separation of motions without employing one of the usually introduced assumptions, we have obtained the equation of 'slow' motion, which is correct for $a < 1$.

2.4. Other variants of solving the equation of 'fast' motion

In Sections 2.1 and 2.3, we have taken into account only the first two terms in solution (7) of the equation of 'fast' motions and stated, that it is justified for $a < 1$. Now we will take into account the first four terms in series (7) and clarify how this will affect the equation of 'slow' motion.

Firstly we present the solution, considering the velocity of 'slow' motion $\dot{X}$ as constant.

For B_1 and B_2, C_1 and C_2, since $\omega \gg 1$, we obtain following expressions:

$$B_1 = -\frac{a}{\omega}\frac{1}{1+\frac{a^2}{8}}\dot{X}, \quad B_2 = \frac{a}{\omega^2}\frac{\beta(64-4a^2)}{(8+a^2)^2}\dot{X}$$
$$C_2 = -\frac{a^2}{\omega}\frac{1}{8+a^2}\dot{X}, \quad C_1 = -\frac{a^2}{\omega^2}\frac{12\beta}{(8+a^2)^2}\dot{X} \tag{24}$$

As follows from formulas (7), (24) we find:

$$a\omega\langle\dot{\psi}\cos\omega t\rangle = a^2\frac{\beta(32-2a^2)}{(8+a^2)^2}\dot{X} \tag{25}$$

As a result, equation of 'slow' motion (5) can be written in the form:

$$\ddot{X} + \beta\left(1-\frac{a^2(32-2a^2)}{(8+a^2)^2}\right)\dot{X} + X = 0 \tag{26}$$

Its solution is still in a poor agreement with solution of Eq. (2) and with results of numerical experiment (see below), though it is always stable. For $a<1$ the difference between coefficients $\dot{X}$ in Eqs. (26) and (11) is less than 25%.

If we take into account higher harmonics in solution (7), for example $\cos 3\omega t$ and $\sin 3\omega t$, it will lead only to modification of coefficient $\dot{X}$ in Eq. (26) of 'slow' motion.

The solution of Eq. (6), without considering the velocity of 'slow' motion $\dot{X}$ as constant and taking into account the first two harmonics, was also conducted. As a result, we obtained equation of 'slow' motion in the form:

$$\ddot{X} + \beta\dot{X} + \frac{(8+a^2)^2}{3a^4-16a^2+64}X = 0 \tag{27}$$

Taylor series expansion of the function $I_0^2(a)$ with order a^4 term retained, yields $I_0^2(a) = 1+(a^2/2)+(3a^4/32)+O(a^6)$. The same operation with the X coefficient in Eq. (27) gives exactly the same expression. For $a<1$ the difference between coefficients X in Eqs. (27) and (23) is less than 20%.

If we take into account higher harmonics in solution (7) it will lead to further approaching of coefficient X in equation of 'slow' motion to $I_0^2(a)$.

3. The range of parameters, in which the solution of the equation of Fidlin oscillator is stable. Reference to numerical experiments

In his paper [5] Fidlin has concluded, that the equation of 'slow' motion (2) corresponds to Eq. (1) for the range of parameters: a and β are of order one, whereas the frequency is much larger than one $\omega \gg 1$. However, it is not always true. For instance, when a=3 and ω=201/s the equation of 'slow' motion (2) does not correspond to Eq. (1), because the solution of Eq. (1) is unstable (this result can be easily obtained in a numerical experiment). The range of parameters, in which the equation of 'slow' motion (2) indeed corresponds to Eq. (1), will be determined in this section.

In fact, the equation of 'slow' motion (2) does not correspond to Eq. (1) if Fidlin's asymptotic approach and the method of direct separation of motions cannot be applied for solving Eq. (1). To define these cases, verification of the fulfillment of the main assumption of vibrational mechanics is provided in this section.

As is noted in Section 2.2, verification of the fulfillment of the main assumption of vibrational mechanics is a posteriori. In this section, to conduct it we use the solution obtained by Fidlin in paper [5]. The solution, obtained by the method of direct separation of motions in Section 2, is not used.

Expression (12) for the ratio $(\dot{X}/X_0)/(\dot{\psi}/\psi_0)$ remains valid for Fidlin's solution. Using Eq. (2) we obtain the following expression for the frequency of 'slow' free vibrations:

$$k = \sqrt{I_0^2(a)-\beta^2/4} \tag{28}$$

Then the main assumption of vibrational mechanics is transformed to the form:

$$\frac{\sqrt{I_0^2(a)-\beta^2/4}}{\omega} < \varepsilon \ll 1 \tag{29}$$

We choose the parameter ε to be equal to 1/5; then for β=1 inequality (29) defines the area under the curve in Fig. 1. This area defines the range of values of parameters, for which we can apply the method of direct separation of motions or Fidlin's asymptotic approach to solve the problem.

Eq. (1) has been solved numerically and the zone of stability of its zero solution has been found. As it was supposed, the area under the curve in Fig. 1 and obtained zone of stability are almost coinciding.

Therefore, we can conclude, that, if the main assumption of vibrational mechanics is fulfilled, then we can apply the method of direct separation of motions and Fidlin's asymptotic approach for the solution of the problem. In this case, we can consider the equation of 'slow' motion (2) to be correct. If it is not fulfilled, then we cannot apply neither Fidlin's

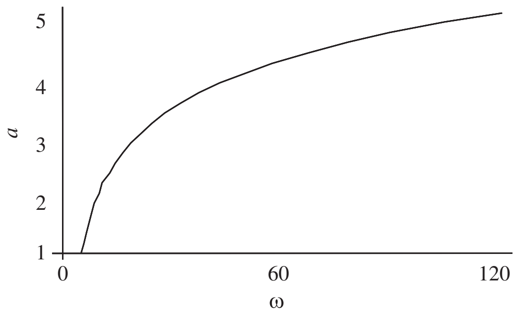

Fig. 1. The boundary of the validity of the main assumption of vibrational mechanics in parameters ω and a (β=1).

asymptotic approach, nor method of direct separation of motions for the solution of Eq. (1). Moreover, in that case, solutions of Eq. (1) are unstable for parameters near the curve shown in Fig. 1. The parametric resonance exists in that case, as for the classical parametric oscillator (see Section 5).

4. Solving the equation of Fidlin oscillator by the method of direct separation of motions in the case of small dissipation coefficient modulations

Here we present the analysis of Eq. (1) in the range of parameters, which differs from one in Section 2. In Eq. (1) we substitute β by $\mu\beta_1$, and $a\omega$ by $\mu\beta_1 b$, where $\mu>0$—small parameter, $\beta_1 \sim O(1)$, $b \sim O(1)$, $\omega \gg 1$. In other words, we consider the case of small dissipation coefficient modulations and equation:

$$\ddot{x}+\mu\beta_1(1-b\cos\omega t)\dot{x}+x=0 \tag{30}$$

for which corresponding equations of 'slow' and 'fast' motions have the form:

$$\ddot{X}+\mu\beta_1(\dot{X}-b\langle\dot{\psi}\cos\omega t\rangle)+X=0 \tag{31}$$

$$\ddot{\psi}+\mu\beta_1(\dot{\psi}-b(\dot{X}+\dot{\psi})\cos\omega t+b\langle\dot{\psi}\cos\omega t\rangle)+\psi=0 \tag{32}$$

4.1. Conventional approximate solution of the equation of 'fast' motions

For the beginning, as in Section 2.1, we apply the method of direct separation of motions in its standard form to solve Eq. (30).

Searching all 2π-periodic solutions of Eq. (32) in form of series

$$\psi=\psi_{01}+\mu\psi_1+\cdots, \tag{33}$$

we obtain, that ψ_{01}=0, and for ψ_1 the following equation is correct:

$$\ddot{\psi}_1+\psi_1=\beta_1 b\dot{X}\cos\omega t, \tag{34}$$

periodic solution of Eq. (34) with constant ('frozen') $\dot{X}$ have the form:

$$\psi_1=B_1\cos\omega t \tag{35}$$

For B_1 we obtain the following expression:

$$B_1=\frac{1}{(1-\omega^2)}\beta_1 b\dot{X} \tag{36}$$

For $\omega \gg 1$ expression (36) can be simplified as follows:

$$B_1=-\frac{1}{\omega^2}\beta_1 b\dot{X} \tag{37}$$

To obtain the equation of 'slow' motion from (31) it is sufficient to determine the expression $\mu\beta_1 b\langle\dot{\psi}\cos\omega t\rangle$. Using formulas (33), (35) and (37) and taking into account that $\langle\sin 2\omega t\rangle$=0, we find:

$$\mu\beta_1 b\langle\dot{\psi}\cos\omega t\rangle=0 \tag{38}$$

So we obtain the equation of 'slow' motion (31) in the following form:

$$\ddot{X}+\mu\beta_1\dot{X}+X=0 \tag{39}$$

As is seen, small dissipation coefficient modulations, under the influence of the employed assumptions, do not affect the resulting equation of 'slow' motion. However, the obtained Eq. (39) is incorrect; this can be confirmed by numerical experiments.

4.2. More accurate solution with dependence of 'fast' motion on 'slow' time taken into account

Now for solving Eq. (30) we apply the advanced method of direct separation of motions, without the assumption that then solving the equation of 'fast' motions we can regard 'slow' variables as constants. As before, we search 2π-periodic solutions of Eq. (32) in form of series (33). As a result we obtain, that $\psi_{01}=0$, and ψ_1 is determined from Eq. (34). We search periodic solution of Eq. (34) in the form

$$\psi_1 = B_1 \cos\omega t + B_2 \sin\omega t \tag{40}$$

where B_1 and B_2 are 'slow' time depending functions, for which we get following expressions:

$$B_1 = -\frac{1}{\omega^2}\beta_1 b\dot{X}, \quad B_2 = \frac{2}{\omega^3}\beta_1 b\ddot{X} \tag{41}$$

but then

$$\mu\beta_1 b\langle\dot{\psi}\, cos\,\omega t\rangle = \frac{\mu^2}{2}\frac{\beta_1^2 b^2}{\omega^2}\ddot{X}$$

Finally, we obtain equation of slow motion (31) in the form:

$$\left(1-\frac{\mu^2}{2}\frac{\beta_1^2 b^2}{\omega^2}\right)\ddot{X}+\mu\beta_1\dot{X}+X=0 \tag{42}$$

Modulations of small dissipation coefficient lead to the reducing of the effective mass in resulting equation of 'slow' motion.

As in Section 2, we can conclude, that, to solve this problem by the method of direct separation of motion, it is necessary to abandon the assumption that 'slow' variables can be considered as constants then solving the equation of 'fast' motions.

5. The solution of the classical equation with oscillating stiffness coefficient by the method of direct separation of motions

Numerous applied problems, analyzed before [7,8], did not require the dependence of 'fast' component on 'slow' time to be taken into account while solving corresponding equations by the method of direct separation of motions. In contrast, the case of Fidlin oscillator makes it necessary. In this section, we define general conditions, when the advanced formulation of the method should be employed. For this purpose we consider the classical equation with oscillating stiffness coefficient.

We examine the following equation, assuming that parameters a and β are of order of one, whereas the frequency is much larger than one:

$$\ddot{x}+\beta\dot{x}+x = a\omega x\cos\omega t \tag{43}$$

Applying the method of direct separation of motions, we obtain the following equations of 'slow' and 'fast' motions corresponding:

$$\ddot{X}+\beta\dot{X}+X = a\omega\langle\psi\cos\omega t\rangle \tag{44}$$

$$\ddot{\psi}+\beta\dot{\psi}+\psi = a\omega(X\cos\omega t+\psi\cos\omega t-\langle\psi\cos\omega t\rangle) \tag{45}$$

The periodic solution of Eq. (45) has the form of the infinite series (7). We take into account only the first two terms in this solution; this is justified for $a<1$. Then for B_1 and B_2 we obtain expressions

$$B_1 = \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2\beta^2}a\omega X, \quad B_2 = \frac{\omega\beta}{(1-\omega^2)^2+\omega^2\beta^2}a\omega X \tag{46}$$

Hence, using the condition $\omega \gg 1$, we obtain

$$B_1 = -\frac{a}{\omega}X, \quad B_2 = \frac{a\beta}{\omega^2}X \tag{47}$$

So, the equation of 'slow' motion (44) appears in the form:

$$\ddot{X}+\beta\dot{X}+\left(1+\frac{a^2}{2}\right)X=0 \tag{48}$$

Now we check whether it is necessary to reject conventional assumptions for solving this classical equation by the method of direct separation of motions, or it can be applied in its traditional form.

As before, while solving Eq. (45) we take into account only the first two terms in (7); this is justified for $a<1$. Now in the Eq. (45) we do not consider the 'slow' variable X to be constant ('frozen'), so B_1 and B_2—are some 'slow'-time t depending functions. Then we obtain

$$B_1 = -\frac{a}{\omega}X, \quad B_2 = \frac{a}{\omega^2}(2\dot{X}+\beta X) \tag{49}$$

To obtain the equation of ‘slow’ motion it is sufficient to determine the expression:

$$\langle \psi \cos\omega t \rangle = \frac{1}{2} B_1 \tag{50}$$

However, the formula (49) for the amplitude B_1 coincides with the formula (47) obtained before. Therefore, the dependence of ‘slow’ variables on time in the equation of ‘fast’ motions does not produce any changes in equation of ‘slow’ motions, i.e. for the solution of the classical equation with oscillating stiffness coefficient method of direct separation of motion can be applied in its conventional form.

The dependence of ‘slow’ variables on time in solving equation of ‘fast’ motion yields only high-order corrections in the amplitudes of ‘fast’ motion. So, it is necessary only in the case, then terms, containing these amplitudes, play the main role in resulting equation of ‘slow’ motion.

6. The comparison of the influence of dissipation and stiffness coefficients modulations on the stability of zero solutions of corresponding equations for various ranges of parameters

6.1. Small dissipation and stiffness coefficients modulations

Firstly, we consider the following classical equation with stiffness coefficients modulations:

$$\ddot{x} + \beta x + \mu(\mu c_1 - d\cos\omega t)x = 0 \tag{51}$$

where $\mu > 0$—small parameter, β, c_1 and d are of order of one.

In this case equations of ‘slow’ and ‘fast’ motions have the form:

$$\ddot{X} + \beta\dot{X} + \mu^2 c_1 X - \mu d \langle \psi \cos\omega t \rangle = 0 \tag{52}$$

$$\ddot{\psi} + \beta\dot{\psi} + \mu^2 c_1 \psi = \mu d(X + \psi)\cos\omega t - \mu d \langle \psi \cos\omega t \rangle) \tag{53}$$

We search for all 2π-periodic solutions of Eq. (53) in the form of series

$$\psi = \psi_{01} + \mu\psi_1 + \cdots, \tag{54}$$

and obtain ψ_{01}=0. The function ψ_1 should be found from the following equation:

$$\ddot{\psi}_1 + \beta\dot{\psi}_1 = \mathrm{d}X\cos\omega t \tag{55}$$

Periodic solution with respect to ωt of this equation with ‘frozen’ X is

$$\psi_1 = -X\frac{d}{\omega^2 + \beta^2}\cos\omega t + X\frac{\beta}{\omega}\frac{d}{\omega^2 + \beta^2}\sin\omega t \tag{56}$$

Taking into account expressions (54) and (56) we obtain equation of ‘slow’ motions in the form:

$$\ddot{X} + \beta\dot{X} + \mu^2\left(c_1 + \frac{d^2}{2(\omega^2 + \beta^2)}\right)X = 0 \tag{57}$$

Thereby, in this case modulations in the stiffness coefficient lead to the increase in the effective stiffness of the system, and do not affect the dissipation coefficient. As a result, for $c_1 < 0$, in the absence of vibration the motion of mechanical system, described by Eq. (51), is unstable, but the presence of oscillations produces the stabilization effect. This result corresponds to the classical problem of stability of the upper (‘overturned’) position of a pendulum under the effect of vibration of its bracket axis (Stephenson–Kapitza pendulum; see, for example, [7,8]).

Now we find out whether the similar effect exists in the system, described by Eq. (1). In Section 4, we have already examined the case of small dissipation coefficient modulations (Eq. (30)). Here we find out, can this modulation produce the stabilization effect in a system, which is unstable without vibration.

First, we consider the equation

$$\ddot{x} + \mu\beta_1(1 - b\cos\omega t)\dot{x} + \mu^2 c_1 x = 0 \tag{58}$$

where $\mu > 0$—small parameter, c_1, β_1 and b—are of order of one, and still $\omega \gg 1$.

We apply the method of direct separation of motions to solve Eq. (58). The dependence of ‘slow’ variables on time is taken into account in solving the equation of ‘fast’ motions. It yields the equation of the ‘slow’ motion in the form:

$$\left(1 - \frac{\mu^2}{2}\frac{\beta_1^2 b^2}{\omega^2}\right)\ddot{X} + \mu\beta_1\dot{X} + \mu^2 c_1 X = 0 \tag{59}$$

As is seen from Eq. (59), this modulation do not affect the stability of corresponding solution, because $\mu^2/\omega^2 \ll 1$.

It seems to be interesting to analyze the following equation:

$$\mu^2 m\ddot{x}+\mu\beta_1(1-b\cos\omega t)\dot{x}+x=0 \tag{60}$$

here $m\sim O(1)$, all other parameters are the same as before. However, this equation is singularly perturbed, because it contains a small coefficient in the term $\ddot{X}$, and so it deserves more detailed consideration, which lies beyond the scope of the present paper.

6.2. Strong dissipation and stiffness coefficients modulations

Now we analyze the influence of strong stiffness and dissipation coefficients modulations on the stability of solution x=0.

To describe the influence of strong modulations of the stiffness coefficient on the stability, it is sufficient to consider the equation:

$$\ddot{x}-kx=a\omega x\cos\omega t \tag{61}$$

where $k>0$. As a result, we obtain that, though the solution of Eq. (61) without modulation is unstable, in some situations it can become stable due to the effect of modulations. This result is easily obtained by the method of direct separation of motions. Searching the solution of the equation of 'fast' motions in the form (7) and taking into account only the first two terms in it (it is justified for $a<1$), we obtain the following equation of 'slow' motions:

$$\ddot{X}+\left(\frac{a^2}{2}-k\right)X=0 \tag{62}$$

For $a^2>2k$ the zero solution of Eq. (62) is stable. Solving the equation of 'fast' motions more accurately and taking into account the first two harmonics, we get the equation of 'slow' motion in the form:

$$\ddot{X}+\left(\frac{4a^2}{8+a^2}-k\right)X=0 \tag{63}$$

For $(4a^2/(8+a^2))>k$ the zero solution of Eq. (63) is stable.

Now we clarify whether the similar effect exists in the system, described by Eq. (1) or resembling, i.e. we consider the equation:

$$\ddot{x}-kx=a\omega\dot{x}\cos\omega t \tag{64}$$

where again $k>0$. We apply the method of direct separation of motions for the solution of Eq. (64), taking into account the dependence of 'slow' variables on time, when solving the equation of 'fast' motions. Searching the solution of the equation of 'fast' motions in the form (7) and taking into account only the first harmonic (it is justified for $a<1$), we obtain the following equation of 'slow' motion:

$$\left(1-\frac{a^2}{2}\right)\ddot{X}-kX=0 \tag{65}$$

or

$$\ddot{X}-\frac{2k}{2-a^2}X=0 \tag{66}$$

The solution of Eq. (66) is always unstable for $a<1$.

Now, we solve the equation of 'fast' motions more accurately, taking into account the first two harmonics in its solution. Corresponding equation of 'slow' motion have the form:

$$\left(1-\frac{2a^2(16-a^2)}{(8+a^2)^2}\right)\ddot{X}-kX=0 \tag{67}$$

or

$$\ddot{X}-\frac{(8+a^2)^2}{3a^4-16a^2+64}kX=0 \tag{68}$$

Expression $((8+a^2)^2)/(3a^4-16a^2+64)$ is positive for all magnitudes of the parameter a, this is why the zero solution of Eq. (68) is always unstable. If we take into account higher harmonics in the solution of the equation of 'fast' motions it will lead to further approaching of the coefficient X in the equation of 'slow' motion to$-kI_0^2(a)$, so its zero solution will always be unstable.

Thereby, we can conclude, that both small and strong dissipation coefficient modulations do not influence the stability of the zero solution.

7. On possible applications of Fidlin's equation

In this section, we demonstrate several model mechanical systems which are described by Eq. (1).

7.1. Oscillations of a rigid body on a rough surface exposed to two-frequency excitation

We consider an one degree of freedom system shown in Fig. 2 and account for the dry friction between the rigid body and the rough surface. Horizontal high-frequency force $F_\Omega \cos(\Omega t+\varepsilon)$ and vertical force $F_\omega \cos\omega t$, which frequency ω is much smaller than frequency Ω, are acting at the body. We can replace the force $F_\omega \cos\omega t$ by the base excitation and assume, that the surface perform oscillations $A_\omega \cos\omega t$ in vertical direction, where $A_\omega = F_\omega/(m\omega^2)$. As shown in [7,8], the dry friction force between the body and the surface transforms into the 'slow' viscous friction force due to vibration with high frequency Ω. This transformation, since $\Omega \gg \omega$, will take place also with respect to force $F_\omega \cos\omega t$. We assume

$$F_\Omega \geqslant f(mg+F_\omega) \tag{69}$$

Then the viscous friction coefficient is determined by the formula [7]

$$\beta = \frac{2}{\pi} f \frac{(mg-F_\omega\cos\omega t)m\Omega}{F_\Omega} = \beta_1-\beta_2\cos\omega t \tag{70}$$

Here g is the gravity acceleration, f is the dry friction coefficient, and

$$\beta_1 = \frac{2}{\pi} f \frac{m^2 g\Omega}{F_\Omega}, \quad \beta_2 = \frac{2}{\pi} f \frac{F_\omega m\Omega}{F_\Omega} \tag{71}$$

As a result the motion of the body is described by the equation:

$$m\ddot{x}+[\beta_1-\beta_2\cos\omega t]\dot{x}+cx=0 \tag{72}$$

Eq. (72) can be transformed to the form:

$$\ddot{x}+\beta_{01}\dot{x}+\lambda^2 x=\beta_{02}\dot{x}\cos\omega t \tag{73}$$

where $\lambda^2 = c/m$, $\beta_{01} = \beta_1/m = (2/\pi)fmg\Omega/F_\Omega$, $\beta_{02} = \beta_2/m = (2/\pi)fF_\omega\Omega/F_\Omega$.

The form of Eq. (73) is similar to Fidlin's Eq. (1). However, we assume that the contact between the body and the surface is permanent, $F_\omega < mg$ and, therefore, $\beta_1 > \beta_2$. It implies that the $\dot{x}$ coefficient in Eq. (73) is always positive, in contrast to Eq. (1), in which this coefficient is negative for some periods of time.

Firstly, we consider Eq. (73) in the case of strong excitation, when $\beta_{02}=O(\omega)$ and $\lambda^2=O(1)$. In this case $\beta_{01}=O(\omega)$, because $\beta_{01} > \beta_{02}$. Searching the solutions of Eq. (73) in the form of series (7) and taking into account only the first harmonic in it, for B_1 and B_2 we obtain expressions (8), in which a is substituted by β_{02}/ω, and β by β_{01}. But then the amplitudes B_1 and B_2 are magnitudes of the same order: $B_1 = B_2 = O(\dot{X}/\omega)$. Therefore, if we take into account the dependence of 'fast' variable on 'slow' time in corresponding equation of 'fast' motions, then it will lead only to insignificant change of the amplitudes B_1 and B_2. So in this case it is not necessary. Equation of 'slow' motion corresponding to Eq. (73) has the form (11), i.e. it differs substantially from Eq. (2).

Nevertheless, in the case of small dissipation coefficient modulations, when $\beta_{01}=\beta_{02}=O(\mu)$, $\lambda^2=O(1)$, where $\mu > 0$—small parameter, Eq. (73) is equivalent to Eq. (30). So, equation of 'slow' motion corresponding to Eq. (73) has the form

$$\left(1-\frac{\beta_{02}^2}{2\omega^2}\right)\ddot{X}+\beta_{01}\dot{X}+\lambda^2 X=0 \tag{74}$$

7.2. Washer, sliding on a rough surface under the effect of vibration

We consider a rigid rod with a washer of mass m on it, lying on a surface (x, y), as is shown in Fig. 3. The friction between the washer and the rod is negligibly small, i.e. we assume, that the washer can move freely along the rod. The washer is connected to the mobile base by a spring with the stiffness c. The rod with the washer and the base is moving along the axis x with some speed $\dot{x}$. The force N acts on the washer to press it to the surface (x, y). As a result, the dry friction force between the washer and the surface is generated. So, the motion of the washer along the axis y is governed by the

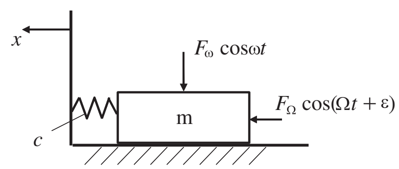

Fig. 2. Oscillations of a rigid body on a rough surface exposed to two-frequency excitation.

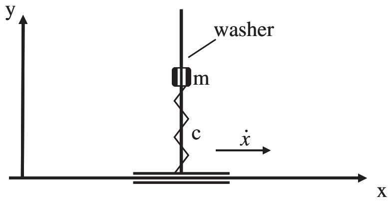

Fig. 3. Washer, sliding on a rough surface under the effect of vibration.

equation:

$$m\ddot{y}+R_y+cy=0 \tag{75}$$

where R_y is the dry friction force, acting along the axis y. It is defined as

$$R_y=\frac{\dot{y}}{V}fN \tag{76}$$

here f is the dry friction coefficient, $V=\sqrt{\dot{x}^2+\dot{y}^2}$ is the magnitude of the velocity of the washer on surface.

We assume that $\dot{x} \gg \dot{y}$, so that Eq. (75) can be written in the form:

$$m\ddot{y}+\frac{\dot{y}}{|\dot{x}|}fN+cy=0 \tag{77}$$

Further we consider two different cases. In the first case we assume, that velocity of the washer along axis x is given in the form:

$$\dot{x}=u+u_1\cos\omega t \tag{78}$$

where $u \gg u_1$, $u>0$, $u_1>0$. So we obtain

$$\frac{1}{|\dot{x}|}=\frac{1}{u}-\frac{u_1}{u^2}\cos\omega t \tag{79}$$

and Eq. (79) transforms to the form

$$m\ddot{y}+\frac{fN}{u}\dot{y}+cy=\frac{fNu_1}{u^2}\dot{y}\cos\omega t \tag{80}$$

Eq. (80) is similar to Eq. (1). However, in this case, the $\dot{y}$ coefficient, again, in contrast to Eq. (1), is always positive. Equation of 'slow' motions (74) corresponds to Eq. (80) with $\lambda^2=c/m=O(1)$, $\beta_{01}=f(N/(um))=O(\mu)$, $\beta_{02}=f(Nu_1/(u^2m))=O(\mu^2)$. In this case, $\beta_{02} \ll \beta_{01}$, because $u \gg u_1$.

In the second case we assume, that the pressing force N have the form:

$$N=N_0-N_1\cos\omega t \tag{81}$$

and the velocity of the rod with the washer along the axis x is constant $\dot{x}=u$. Then Eq. (77) transforms to the form

$$m\ddot{y}+\frac{fN_0}{u}\dot{y}+cy=\frac{fN_1}{u}\dot{y}\cos\omega t \tag{82}$$

It is similar to Eq. (1), but in this case the $\dot{y}$-coefficient again is always positive, because the condition $N_1<N_0$ ensures that the washer is never detached from the surface (x, y). Equation of 'slow' motion (74) with $\lambda^2=c/m=O(1)$, $\beta_{01}=f(N_0/(um))=O(\mu)$, $\beta_{02}=f(N_1/(um))=O(\mu)$ corresponds to Eq. (82); as in Section 7.1, $\beta_{01}>\beta_{02}$.

The system considered in this section, can be used, for example, for modeling a car brake system.

We note that the similar equation governs the motion of a solid body at the plane rough surface with a double slope to the direction of vibration [12, pp. 300–303].

We also take notice of the fact, that in all considered cases vibrational or 'kinematic' transformation of the dry friction to the viscous friction results in the modulation of the dissipation coefficient. We emphasize that in these examples, in contrast to Eq. (1), these modulations are constrained in the sense that the dissipation coefficient remains positive. In all likeness, this constrain may be withdrawn for mechanical systems with 'descending' dry friction characteristic, and also for systems relevant to electro- and radio-techniques.

7.3. Linear oscillator with mass modulation

Another interesting system, the equation of motion of which can be similar to Fidlin's Eq. (1), is a linear oscillator with time-varying mass [13]. Mass variations can be caused, for example, by water (rain) drops hitting the oscillator. The equation of motion of this oscillator has the form [13]:

$$M\ddot{y}=\dot{M}(w-\dot{y})-cy+f \tag{83}$$

here $y=y(t)$ is the displacement of the oscillator, $M=M(t)$ is its time-depending mass, w is the velocity of the drops hitting the oscillator, c is the stiffness of a spring, f is an external force.

We assume that oscillator mass variations occur harmonically:

$$M = M_0 + m\cos\omega t \tag{84}$$

Then Eq. (83) takes the form

$$M_0\ddot{y} + cy = -m(w-\dot{y})\omega\sin\omega t - m\ddot{y}\cos\omega t + f \tag{85}$$

If we assume the amplitude m of the mass variations to be negligibly small with respect to constant component M_0, then Eq. (85) can be written in the form:

$$M_0\ddot{y} - m\omega\dot{y}\sin\omega t + cy = F \tag{86}$$

where F is total external force. The homogeneous part of Eq. (86) coincides with Eq. (1).

As appears from this section, Eq. (1) can be perceived as a model for systems with oscillating inertia properties. In addition, an analogy between a system with modulated mass and a system with modulated stiffness, in which the changing of the effective stiffness is well-known (see Sections 5 and 6), can be easily drawn. Thus, the increase in the effective stiffness produced due to the dissipation coefficient modulation might be explained by drawing an analogy between these systems.

However, in our opinion, this explanation is questionable. The equation of motion of linear oscillator with time-varying mass (83) is similar to the equation with dissipation modulation only because of the presence of the term $\dot{M}(w-\dot{y})$ in it. If this term was absent, then the discussed analogy could not been drawn. Therefore, in our opinion, a system with oscillating mass cannot be considered as equivalent to a system with oscillating dissipation term. Moreover, in the case of strong excitation the analogy between a system with modulated mass and a system with modulated stiffness is non-convincing.

In our opinion, the physical explanation of the effect that dissipation coefficient modulation produces the increase in the effective stiffness deserves detailed discussion.

8. On the comparison of the method of direct separation of motions with asymptotic methods

One of the main results of the paper is the following: the method of direct separation of motions, at least in terms of Eq. (1), leads to the same results as Fidlin's asymptotic approach. But the comparison of these methods, in terms of accuracy, simplicity, and generality is not a part of the scope of the present paper. Apparently, Fidlin's asymptotic approach is more complicated then the method of direct separation of motions, and the range of application of the method of direct separation of motions, with the dependence ψ on t being taken into account, is not narrower, then of Fidlin's approach.

The substantiation of the method of direct separation of motions, based on the rigorous theorems of N.N. Bogolubov, V.M. Volosov and V.I. Morgunov, is given in the book [7]. The general comparison of the method of direct separation of motions with asymptotic methods, particularly with the multiple scales perturbation method, is provided in the book [8]. The main advantages of the method of direct separation of motions over asymptotic methods can be listed as follows:

1. The transformation to the canonical equations, i.e. equations of the first order, is not required.
2. Resulting equations of 'slow' motions have the form of dynamic equations. So the important classes of so-called potential on the average dynamical systems, vibratory-smooth systems, systems with 'fast' generalized coordinates (in which the order of equations of 'slow' motions is less than the order of initial equations), can be easily marked out.
3. The method has simple physical interpretation on each step.
4. All simplifications are concentrated in solving corresponding equations of 'fast' motions. These equations can be solved approximately, without bringing in essential error in equations of 'slow' motions, because 'fast' motions are introduced into equations of 'slow' motions under the sign of averaging.

All these advantages (as well as shortcomings) are described in detail in the book [7].

Certainly, the solution of corresponding equation of 'fast' motions by the method of direct separation of motions become more complicated with the dependence of 'fast' variable on 'slow' time being taken into account. However, all the main advantages of the method of direct separation of motions over asymptotic methods, particularly over Fidlin's approach, are preserved in the cases, when this challenge has to be met.

9. Conclusions

Linear differential equation of the second order with modulation of the damping coefficient ('Fidlin oscillator') was considered in the present paper. This equation was analyzed in different ranges of parameters, in contrast to book [6]. The findings reported in this paper are summarized as follows:

- It is shown that the method of direct separation of motions can be applied for solving problems concerned with velocity-dependent high frequency excitation as exemplified in the case of the Fidlin oscillator.

- To capture dynamics of the Fidlin oscillator, the method of direct separation of motions should be modified. Specifically, the equation of ‘fast’ motions should be solved more accurately, with the dependence of it solution on ‘slow’ time being taken into account.
- The remarkable, and not yet completely understood from the physical point of view, feature of the Fidlin oscillator is that the modulation of the dissipation coefficient produces the increase of the effective stiffness of the system with respect to ‘slow’ loads. However, equally strong high-frequency modulation of the stiffness of the system produces just the increase of the effective stiffness of the system, while the magnitude of the dissipation coefficient is unaffected.
- Although the high frequency stiffness modulations can stabilize an unstable position of a mechanical system (Stephenson–Kapitza pendulum), it is not the case with the similar modulation of the dissipation coefficient. This statement holds true both in the case, when the instability is triggered by the negative dissipation coefficient, and in the cause, when the instability is triggered by the negative stiffness coefficient.
- In the situations, when the main assumption of the vibrational mechanics is not fulfilled, modulations of the stiffness coefficient as well as of the dissipation coefficient can lead to the instability of the system (parametric resonance).
- Several mechanical systems, modeled by the Fidlin oscillator, are introduced and discussed.

As a part of conclusions, the Appendix summarizes the considered original equations and the equations of ‘slow’ motion derived in various assumptions. The ranges of parameters for which these equations are valid are also presented.

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Appendix. The considered equations, the ranges of parameters and the resulting equations of ‘slow’ motion in various assumptions (MDSM—method of direct separation of motions)

The considered equation: $\ddot{x}+\beta\dot{x}+cx=A(\dot{x},x)\cos\omega t$; $\omega \gg 1$			
The resulting equation of ‘slow’ motions: $M\ddot{X}+B\dot{X}+CX=0$			
Values of the coefficients β, c and of the function $A(x,\dot{x})$ in the considered equation	Ranges of the parameters	Assumptions employed then solving corresponding equation of ‘fast’ motions by the MDSM ($x=X(t)+\psi(t,\omega t)$)	Values of the coefficients M, B and C in corresponding equation of ‘slow’ motion
$c=1$, $A(x,\dot{x})=a\omega\dot{x}$	a, $\beta-O(1)$	$a<1$ $\dot{X}=\text{const}$ $a<1$ $\dot{X}\neq\text{const}$ $1<a<1.5$ $\dot{X}=\text{const}$ $1<a<1.5$ $\dot{X}\neq\text{const}$	$M=1$ $B=\beta\left(1-\frac{a^2}{2}\right)$, $C=1$ $B=\beta$, $C=\dfrac{2}{2-a^2}$ $B=\beta\left(1-\dfrac{a^2(32-2a^2)}{(8+a^2)^2}\right)$, $C=1$ $B=\beta$, $C=\dfrac{(8+a^2)^2}{3a^4-16a^2+64}$
$\beta=\mu\beta_1$, $c=1$, $A(x,\dot{x})=\mu\beta_1 b\dot{x}\cos\omega t$	$0<\mu\ll 1$; b, $\beta_1-O(1)$	$\dot{X}=\text{const}$ $\dot{X}\neq\text{const}$	$C=1$, $B=\mu\beta_1$ $M=1$ $M=1-\dfrac{\mu^2}{2}\dfrac{\beta_1^2 b^2}{\omega^2}$
$c=1$, $A(x,\dot{x})=a\omega x$	$a,\beta-O(1)$	$a<1$ $X=\text{const}$ $a<1$ $X\neq\text{const}$	$M=1$, $B=\beta$, $C=1+\frac{a^2}{2}$
$c=\mu^2 c_1$, $A(x,\dot{x})=\mu dx\cos\omega t$ $\beta=\mu\beta_1$, $c=\mu^2 c_1$, $A(x,\dot{x})=\mu\beta_1 b\dot{x}\cos\omega t$	$0<\mu\ll 1$; $c_1,\beta,d-O(1)$	$X=\text{const}$ $\dot{X}\neq\text{const}$	$M=1$, $B=\beta$, $C=\mu^2\left(c_1+\frac{d^2}{2(\omega^2+\beta^2)}\right)$ $M=1-\dfrac{\mu^2}{2}\dfrac{\beta_1^2 b^2}{\omega^2}$, $B=\mu\beta_1$, $C=\mu^2 c_1$
$\beta=0$, $c=-k$, $A(x,\dot{x})=a\omega x$ $\beta=0$, $c=-k$, $A(x,\dot{x})=a\omega\dot{x}$	$a,k-O(1)$; $k>0$	$1<a<1.5$ $X=\text{const}$ $1<a<1.5$ $\dot{X}\neq\text{const}$	$M=1$, $B=0$ $C=\dfrac{4a^2}{8+a^2}-k$ $C=-\dfrac{(8+a^2)^2}{3a^4-16a^2+64}k$

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